

Spaces of matrices with a sole eigenvalue

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Abstract

Let \mathbb{K} be an arbitrary (commutative) field and $\overline{\mathbb{K}}$ be an algebraic closure of it. Let V be a linear subspace of $M_n(\mathbb{K})$, with $n \geq 3$. We show that if every matrix of V has at most one eigenvalue in \mathbb{K} , then $\dim V \leq 1 + \binom{n}{2}$. If every matrix of V has a sole eigenvalue in $\overline{\mathbb{K}}$ and $\dim V = 1 + \binom{n}{2}$, we show that V is similar to the space of all upper-triangular matrices with equal diagonal entries, except if $n = 3$ and \mathbb{K} has characteristic 3, or if $n = 4$ and \mathbb{K} has characteristic 2. In both of those special cases, we classify the exceptional solutions up to similarity.

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1 Introduction

In this article, we let \mathbb{K} be an arbitrary (commutative) field, and we choose an algebraic closure $\overline{\mathbb{K}}$ of it. We denote by $M_n(\mathbb{K})$ the algebra of square matrices with n rows and entries in \mathbb{K} , and by $GL_n(\mathbb{K})$ its group of invertible elements. We also denote by $M_{n,p}(\mathbb{K})$ the vector space of matrices with n rows, p columns and entries in \mathbb{K} . The transpose of a matrix M is denoted by M^T . Two linear subspaces V and W of $M_n(\mathbb{K})$ are called **similar**, and we write $V \sim W$, if $W = PVP^{-1}$ for some $P \in GL_n(\mathbb{K})$ (i.e., V and W represent, in

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a change of basis, the same set of endomorphisms of an n -dimensional vector space). For $M \in M_n(\mathbb{K})$, we denote by $\text{Sp}_{\mathbb{K}}(M)$ its spectrum in \mathbb{K} , i.e., its set of eigenvalues in the field \mathbb{K} , and by $\text{tr}(M)$ its trace.

For $(a_1, \dots, a_n) \in \mathbb{K}^n$, we denote by $\text{Diag}(a_1, \dots, a_n)$ the diagonal matrix of $M_n(\mathbb{K})$ with diagonal entries a_1, \dots, a_n .

Given two integers a and b , we write $a \mid b$ if a divides b , and $a \wedge b = 1$ if a is prime with b .

Linear spaces of square matrices with conditions on their spectrum have been the topic of quite a few papers in the past decades. The first important results can be traced back to Gerstenhaber, who proved the following result in the case $\# \mathbb{K} \geq n$ (see also [3] for a simplified proof and an extension to the case $\# \mathbb{K} \geq 3$, and [9] for a proof with no restriction on the field):

Theorem 1 (Gerstenhaber, Serezhkin). *Let V be a linear subspace of $M_n(\mathbb{K})$ in which every matrix is nilpotent.*

Then $\dim V \leq \binom{n}{2}$. If equality holds, then V is similar to the subspace $\text{NT}_n(\mathbb{K})$ of strictly upper-triangular matrices.

In [4], Omladič and Šemrl consider the following general problem: given $k \in \llbracket 1, n-1 \rrbracket$, determine the maximal dimension for a linear subspace of $M_n(\mathbb{K})$ in which every matrix has at most k eigenvalues in $\overline{\mathbb{K}}$, and classify the subspaces with the maximal dimension. They solved the problem in the special case $\mathbb{K} = \mathbb{C}$, for $k = 1$, $k = 2$ and n odd, and $k = n-1$ with the additional condition that the subspaces under consideration contain a matrix which has exactly $n-1$ distinct eigenvalues in $\overline{\mathbb{K}}$. In the subsequent [2], Loewy and Radwan considered mainly the “upper bound” component of the problem, and extended Omladič and Šemrl’s results to an arbitrary field of characteristic 0, whilst solving the additional cases $k = 3$ and $k = n-1$.

In this paper, we tackle the case $k = 1$ for an arbitrary field and extend the upper bound on the dimension to a larger class of subspaces. Let us start with a few definitions.

Definition 1. Let V be a linear subspace of $M_n(\mathbb{K})$.

We say that V is a **1-spec subspace** when $\# \text{Sp}_{\mathbb{K}}(M) \leq 1$ for every $M \in V$.

We say that V is a **$\overline{1}$ -spec subspace** when $\# \text{Sp}_{\overline{\mathbb{K}}}(M) = 1$ for every $M \in V$.

We say that V has a **trivial spectrum** when $\text{Sp}_{\mathbb{K}}(M) \subset \{0\}$ for every $M \in V$.

We say that V is **nilpotent** when all its elements are nilpotent matrices, i.e., $\text{Sp}_{\overline{\mathbb{K}}}(M) = \{0\}$ for every $M \in V$.

Spaces with a trivial spectrum are linked to the affine subspaces of non-singular matrices of $M_n(\mathbb{K})$. In [5] and [8], two independent proofs are given of the fact that every linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum has a dimension lesser than or equal to $\binom{n}{2}$. In [7], spaces with a trivial spectrum whose dimension reaches this upper bound are classified up to similarity, extending Gerstenhaber's theorem (provided $\# \mathbb{K} > 2$).

Here are our main results:

Theorem 2. *Let V be a 1-spec subspace of $M_n(\mathbb{K})$.*

If $\text{char}(\mathbb{K}) = 2$ and $n = 2$, then $\dim V \leq 3$. Otherwise $\dim V \leq 1 + \binom{n}{2}$.

The following corollary is trivial but must be stated:

Corollary 3. *Let V be a $\overline{1}$ -spec subspace of $M_n(\mathbb{K})$.*

If $\text{char}(\mathbb{K}) = 2$ and $n = 2$, then $\dim V \leq 3$. Otherwise $\dim V \leq 1 + \binom{n}{2}$.

Note that if $\text{char}(\mathbb{K}) \neq 2$ or $n \neq 2$, the subspace $\mathbb{K}I_n + \text{NT}_n(\mathbb{K})$ of all upper triangular matrices with equal diagonal entries is a $\overline{1}$ -spec subspace of dimension $1 + \binom{n}{2}$.

Moreover, setting

$$\mathfrak{sl}_n(\mathbb{K}) := \{M \in M_n(\mathbb{K}) : \text{tr}(M) = 0\},$$

we see that if $\text{char}(\mathbb{K}) = 2$, then a matrix of $M_2(\mathbb{K})$ has exactly one eigenvalue in $\overline{\mathbb{K}}$ if and only if its trace is zero: it follows that $\mathfrak{sl}_2(\mathbb{K})$ is a 3-dimensional $\overline{1}$ -spec subspace of $M_2(\mathbb{K})$. The above upper bounds are therefore tight, and we have the following full description of the $\overline{1}$ -spec subspaces of $M_2(\mathbb{K})$ when $\text{char}(\mathbb{K}) = 2$:

Proposition 4. *If $\text{char}(\mathbb{K}) = 2$, then the $\overline{1}$ -spec linear subspaces of $M_2(\mathbb{K})$ are the linear subspaces of $\mathfrak{sl}_2(\mathbb{K})$.*

We turn to the description of the $\overline{1}$ -spec subspaces of $M_n(\mathbb{K})$ with maximal dimension. Here is the most general situation:

Theorem 5. *Let V be a $\overline{1}$ -spec subspace of $M_n(\mathbb{K})$ such that $\dim V = 1 + \binom{n}{2}$, with $n \geq 3$. If $n \geq 5$ or $\text{char}(\mathbb{K}) \wedge n = 1$, then $V \sim \mathbb{K}I_n + \text{NT}_n(\mathbb{K})$.*

Theorem 6. Assume $\text{char}(\mathbb{K}) = 2$. Then, up to similarity, there are exactly two $\bar{1}$ -spec subspaces of $M_4(\mathbb{K})$ with dimension 7: $\mathbb{K}I_4 + \text{NT}_4(\mathbb{K})$ and the linear subspace

$$\mathcal{H} := \mathbb{K}I_4 + \left\{ \begin{bmatrix} 0 & l_1 & l_2 & x \\ c_2 & 0 & y & c_1 \\ c_1 & x & 0 & c_2 \\ y & l_2 & l_1 & 0 \end{bmatrix} \mid (l_1, l_2, c_1, c_2, x, y) \in \mathbb{K}^6 \right\}.$$

Theorem 7. Assume \mathbb{K} is an algebraically closed field of characteristic 3. Then, up to similarity, there are exactly three $\bar{1}$ -spec subspaces of $M_3(\mathbb{K})$ with dimension 4: $\mathbb{K}I_3 + \text{NT}_3(\mathbb{K})$ and the two subspaces

$$\left\{ \begin{bmatrix} t & x & 0 \\ 0 & t & y \\ z & 0 & t \end{bmatrix} \mid (t, x, y, z) \in \mathbb{K}^4 \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} t & x & z \\ -z & t & y \\ x & 0 & t \end{bmatrix} \mid (t, x, y, z) \in \mathbb{K}^4 \right\}.$$

For an arbitrary field of characteristic 3, the case $n = 3$ is far more complicated: we wait until Section 4 to state the precise results.

We do not know yet how to classify the 1-spec subspaces of $M_n(\mathbb{K})$ with maximal dimension, although the following conjecture seems reasonable and would, if true, solve the question when $\# \mathbb{K} > 2$ and $n > 4$ (using the results of [7]):

Conjecture 1. Let V be a 1-spec subspace of $M_n(\mathbb{K})$ such that $\dim V = 1 + \binom{n}{2}$. Assume $n > 4$. Then there exists a linear subspace W of $M_n(\mathbb{K})$ with a trivial spectrum such that $V = \mathbb{K}I_n + W$.

Proof of Corollary 3 and Theorem 5 in the case $\text{char}(\mathbb{K}) \wedge n = 1$.

Assume $\text{char}(\mathbb{K}) \wedge n = 1$. Then the results from Corollary 3 and Theorem 5 are easy consequences of Gerstenhaber's theorem. Let indeed V be a $\bar{1}$ -spec subspace of $M_n(\mathbb{K})$. Then $W := \text{Ker}(\text{tr}|_V)$ is a nilpotent subspace of $M_n(\mathbb{K})$ (since $\text{char}(\mathbb{K}) \wedge n = 1$) and $\text{codim}_V W \leq 1$. Gerstenhaber's theorem shows that $\dim W \leq \binom{n}{2}$, and hence

$$\dim V = \text{codim}_V W + \dim W \leq 1 + \binom{n}{2}.$$

Assume now that $\dim V = 1 + \binom{n}{2}$. Then W is a hyperplane of V . Therefore $\dim W = \binom{n}{2}$ and $W \sim \text{NT}_n(\mathbb{K})$ (see [9]). Note also that the above inequality

shows that V is maximal among the $\bar{1}$ -spec subspaces of $M_n(\mathbb{K})$. However $\mathbb{K}I_n + V$ is a $\bar{1}$ -spec subspace containing V , which shows that $I_n \in V$. Since $\text{char}(\mathbb{K}) \wedge n = 1$, one has $I_n \notin W$ hence $V = \mathbb{K}I_n + W$, and we deduce that $V \sim \mathbb{K}I_n + \text{NT}_n(\mathbb{K})$. \square

Structure of the article:

The article has two main parts. In Section 2, we prove Theorem 2, using arguments that are very similar to the ones used in Theorem 9 of [8].

The remaining sections deal with the classification of $\bar{1}$ -spec subspaces with maximal dimension. Section 3 is devoted to a proof of Theorems 5 and 6: adapting some ideas of [7], we combine a key lemma (Proposition 8) from Section 2 with Gerstenhaber's theorem in order to sort out the structure of $\bar{1}$ -spec subspaces with maximal dimension. We first work out the case $n \geq 5$, and then the case $n = 4$ and $\text{char}(\mathbb{K}) = 2$. In Section 4, we use the same line of reasoning to solve the case $n = 3$ and $\text{char}(\mathbb{K}) = 3$: in that one, many exceptional solutions arise, and the main difficulty lies in determining necessary and sufficient conditions for two of them to be similar. As we shall see, the classification depends on some arithmetic properties of the field \mathbb{K} .

2 An upper bound for the dimension of a 1-spec subspace

2.1 On the rank 1 matrices in a 1-spec subspace

Notation 2. Let V be a linear subspace of $M_n(\mathbb{K})$, and let $X \in \mathbb{K}^n \setminus \{0\}$. We set $V_X := \{M \in V : \text{Im } M \subset \mathbb{K}X\}$.

Our proofs of Theorems 2 and 5 are based on the following result:

Proposition 8. *Let V be a 1-spec subspace of $M_n(\mathbb{K})$, with $n \geq 2$. Then:*

- (i) *either $n = 2$, $\text{char}(\mathbb{K}) = 2$ and $V = \mathfrak{sl}_2(\mathbb{K})$;*
- (ii) *or there exists $X \in \mathbb{K}^n \setminus \{0\}$ such that $V_X = \{0\}$.*

The proof involves the following result from [8] (Proposition 10):

Lemma 9. *Let V be a linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum. Then there exists $i \in \llbracket 1, n \rrbracket$ such that $V_{e_i} = \{0\}$, where (e_1, \dots, e_n) denotes the canonical basis of \mathbb{K}^n .*

Let us prove a corollary of it:

Corollary 10. *Let V be a linear subspace of $M_n(\mathbb{K})$ with a trivial spectrum. Then there exists a basis (f_1, \dots, f_n) of \mathbb{K}^n such that $V_{f_i} = \{0\}$ for every $i \in \llbracket 1, n \rrbracket$.*

Proof. Denote by F the linear subspace of \mathbb{K}^n spanned by the non-zero vectors X such that $V_X = \{0\}$.

Applying Lemma 9 to all the subspaces that are similar to V shows that every basis of \mathbb{K}^n contains a vector of F . Classically, this shows that $F = \mathbb{K}^n$, which proves the claimed result: indeed, if the contrary holds, then F is included in a linear hyperplane H of E ; since $GL_n(\mathbb{K})$ acts transitively on the set of linear hyperplanes of E , we may find a basis (f_1, \dots, f_n) of \mathbb{K}^n such that H is defined by the equation $x_1 + \dots + x_n = 0$ in this basis, hence none of the vectors f_1, \dots, f_n belongs to F , a contradiction. \square

We may now prove Proposition 8.

Proof of Proposition 8. We assume that $V_X \neq \{0\}$ for every $X \in \mathbb{K}^n \setminus \{0\}$, and prove that (i) holds.

Denote by (e_1, \dots, e_{n-1}) the canonical basis of \mathbb{K}^{n-1} . We naturally identify \mathbb{K}^{n-1} with the subspace $\mathbb{K}^{n-1} \times \{0\}$ of \mathbb{K}^n .

For $(i, j) \in \llbracket 1, n \rrbracket^2$, denote by $E_{i,j}$ the elementary matrix of $M_n(\mathbb{K})$ with entry 1 at the (i, j) -spot and zeroes elsewhere. Denote by W the linear subspace of V consisting of its matrices with zero as the last row, and note that 0 is an eigenvalue of every matrix of W , hence W has a trivial spectrum. For $M \in W$, write

$$M = \begin{bmatrix} K(M) & ? \\ 0 & 0 \end{bmatrix} \quad \text{with } K(M) \in M_{n-1}(\mathbb{K}).$$

Then $K(W)$ is a linear subspace of $M_{n-1}(\mathbb{K})$ with a trivial spectrum. Applying Corollary 10 to $K(W)$, we find $P \in GL_{n-1}(\mathbb{K})$ such that, for $F := PK(W)P^{-1}$, one has $F_{e_i} = \{0\}$ for every $i \in \llbracket 1, n-1 \rrbracket$.

Set $V' := QVQ^{-1}$, where $Q := \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix}$. Then the assumptions show that

$V'_X \neq \{0\}$ for every $X \in \mathbb{K}^n \setminus \{0\}$.

Let $i \in \llbracket 1, n-1 \rrbracket$. Then $V'_{e'_i}$ contains a non-zero matrix M , which must belong to QWQ^{-1} : however $F_{e_i} = \{0\}$ hence the first $n-1$ columns of M are zero, which shows that M is a scalar multiple of $E_{i,n}$. It follows that $E_{i,n} \in V'$ for every $i \in \llbracket 1, n-1 \rrbracket$, hence V' contains $\text{span}(E_{1,n}, \dots, E_{n-1,n})$. Therefore V contains

$\text{span}(E_{1,n}, \dots, E_{n-1,n})$. Conjugating V with arbitrary matrices of $\text{GL}_n(\mathbb{K})$, this generalizes as follows: V contains every rank 1 matrix with zero trace.

Since V is a linear subspace, it must then contain the matrix $A := E_{1,2} + E_{2,1}$, which has $(x^2 - 1)x^{n-2}$ as characteristic polynomial. This polynomial must have at most one root in \mathbb{K} , which shows that $n = 2$ and $\text{char}(\mathbb{K}) = 2$. Thus V contains the basis $(E_{1,2}, E_{2,1}, J)$ of $\mathfrak{sl}_2(\mathbb{K})$, where $J := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, hence $\mathfrak{sl}_2(\mathbb{K}) \subset V \subset M_2(\mathbb{K})$, the last inclusion being sharp since obviously $E_{1,1} \notin V$. Therefore $V = \mathfrak{sl}_2(\mathbb{K})$, which concludes our proof. \square

2.2 Proof of Theorem 2

Let V be a 1-spec subspace of $M_n(\mathbb{K})$.

If $n = 1$, we trivially have $\dim V \leq 1 = 1 + \binom{1}{2}$.

If $n = 2$ and $\text{char}(\mathbb{K}) \neq 2$, then $V \subsetneq M_2(\mathbb{K})$ since V does not contain $\text{Diag}(1, 0)$, and hence $\dim V \leq 3$.

Assume that $n \geq 3$ or that $n = 2$ and $\text{char}(\mathbb{K}) \neq 2$. Then Proposition 8 shows that $V_X = \{0\}$ for some $X \in \mathbb{K}^n \setminus \{0\}$. Conjugating V with a well-chosen non-singular matrix, we lose no generality in assuming that $V_{e_n} = \{0\}$, where $e_n = [0 \ \dots \ 0 \ 1]^T$.

For $M \in V$, denote by $C(M)$ its last column, and set $U := \{M \in V : C(M) = 0\}$. For $M \in U$, write

$$M = \begin{bmatrix} J(M) & 0 \\ ? & 0 \end{bmatrix} \quad \text{with } J(M) \in M_{n-1}(\mathbb{K}).$$

Since $V_{e_n} = \{0\}$, the linear map $M \mapsto J(M)$ is one-to-one on U , hence the rank theorem shows that

$$\dim V \leq n + \dim U = n + \dim J(U).$$

However V is a 1-spec subspace of $M_n(\mathbb{K})$, so, for every $M \in U$, the matrix $J(M)$ cannot have a non-zero eigenvalue in \mathbb{K} . Theorem 9 of [8] then shows that $\dim J(U) \leq \binom{n-1}{2}$. Therefore $\dim V \leq n + \binom{n-1}{2} = 1 + \binom{n}{2}$, as claimed.

Remark 1. Using a similar line of reasoning, it may be proven that $\mathfrak{sl}_2(\mathbb{K})$ is the sole 3-dimensional 1-spec subspace of $M_2(\mathbb{K})$ if $\text{char}(\mathbb{K}) = 2$.

3 On $\bar{1}$ -spec subspaces with maximal dimension

Let $n \geq 2$ and V be a $\bar{1}$ -spec subspace of $M_n(\mathbb{K})$ such that $\dim V = 1 + \binom{n}{2}$. If $\text{char}(\mathbb{K}) \wedge n = 1$, then we already know that Theorem 5 holds for V (see the end of Section 1). In the rest of the proof, we assume that $\text{char}(\mathbb{K}) \mid n$. The case $n = 2$ has already been tackled (see Proposition 4).

From now on, we assume that $n \geq 3$.

Notice that $\mathbb{K}I_n + V$ is a $\bar{1}$ -spec subspace of $M_n(\mathbb{K})$ containing V , hence $V = \mathbb{K}I_n + V$ by Corollary 3, and in particular $I_n \in V$.

Finally, we will need the following notation and the subsequent remarks:

Notation 3. Given $M \in M_n(\mathbb{K})$, with characteristic polynomial $x^n + \sum_{k=0}^{n-1} a_k x^k \in \mathbb{K}[x]$, we set $c_k(M) := (-1)^k a_{n-k}$ for every $k \in \llbracket 1, n \rrbracket$.

Classically, c_2 is a quadratic form on $M_n(\mathbb{K})$ and its polar form b_2 , defined as $b_2(A, B) := c_2(A + B) - c_2(A) - c_2(B)$, satisfies:

$$\forall (A, B) \in M_n(\mathbb{K})^2, \quad b_2(A, B) = \text{tr}(A) \text{tr}(B) - \text{tr}(AB).$$

Since $\text{char}(\mathbb{K}) \mid n$, every matrix of V has trace 0, therefore

$$\forall (A, B) \in V^2, \quad b_2(A, B) = -\text{tr}(AB).$$

Notice that every singular matrix of V is automatically nilpotent, which leads to the following result:

Lemma 11. *Let $(A, B) \in V^2$. Assume that A , B and $A + B$ are singular. Then $\text{tr}(AB) = 0$.*

3.1 Setting things up

We start with the same line of reasoning as in Section 2. Since $n \geq 3$, Proposition 8 shows that we may replace V with a similar subspace so as to have, for $e_n = [0 \ \cdots \ 0 \ 1]^T$,

$$V_{e_n} = \{0\}. \tag{1}$$

For $M \in V$, denote by $C(M)$ its last column, and set $Z := \{M \in V : C(M) = 0\}$. For $M \in Z$, write

$$M = \begin{bmatrix} J(M) & 0 \\ ? & 0 \end{bmatrix} \quad \text{with } J(M) \in M_{n-1}(\mathbb{K}).$$

Note that $J(Z)$ is a nilpotent subspace of $M_{n-1}(\mathbb{K})$. Since $V_{e_n} = \{0\}$, the map J is one-to-one. The rank theorem shows that $\dim V = \dim C(V) + \dim Z = \dim C(V) + \dim J(Z)$. However $\dim C(V) \leq n$ and $\dim J(Z) \leq \binom{n-1}{2}$ whilst $\dim V = n + \binom{n-1}{2}$. Therefore

$$\dim C(V) = n \quad \text{and} \quad \dim J(Z) = \binom{n-1}{2}.$$

Applying Gerstenhaber's theorem to $J(Z)$, we obtain $P \in GL_{n-1}(\mathbb{K})$ such that $PJ(Z)P^{-1} = NT_{n-1}(\mathbb{K})$. Set then $P_1 := \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \in GL_n(\mathbb{K})$ and replace V with $P_1VP_1^{-1}$. Notice that $P_1^{-1}e_n = e_n$, therefore condition (1) is still satisfied in this new setting, but we now have

$$J(Z) = NT_{n-1}(\mathbb{K}). \tag{2}$$

Denote now by W the set of all matrices of V with zero as first row. For $M \in W$, write

$$M = \begin{bmatrix} 0 & 0 \\ ? & R(M) \end{bmatrix} \quad \text{with } R(M) \in M_{n-1}(\mathbb{K}).$$

Note the following obvious properties of $R(W)$:

- (i) $R(W)$ is a nilpotent subspace of $M_{n-1}(\mathbb{K})$;
- (ii) The shape of $J(Z)$ shows that, for every $N \in NT_{n-2}(\mathbb{K})$, the subspace $R(W)$ possesses a matrix of the form $\begin{bmatrix} N & 0 \\ ? & 0 \end{bmatrix}$.

Let $C_1 \in M_{n-2,1}(\mathbb{K})$. Since $\dim C(V) = n$, we know that V contains a matrix of the form $\begin{bmatrix} ? & ? & 0 \\ ? & ? & C_1 \\ ? & ? & 0 \end{bmatrix}$. By adding a well-chosen matrix of Z and a scalar

multiple of I_n , we deduce that W contains a matrix of the form $\begin{bmatrix} 0 & 0 & 0 \\ ? & ? & C_1 \\ ? & ? & ? \end{bmatrix}$,

i.e., $R(W)$ contains a matrix of the form $\begin{bmatrix} ? & C_1 \\ ? & ? \end{bmatrix}$. Since this holds for every $C_1 \in M_{n-2,1}(\mathbb{K})$, combining this with point (ii) above yields that $\dim R(W) \geq (n-2) + \binom{n-2}{2} = \binom{n-1}{2}$, and hence $\dim R(W) = \binom{n-1}{2}$ by Gerstenhaber's theorem. Since $R(W)$ is nilpotent, no matrix of it has $[0 \ \cdots \ 0 \ 1]^T$ as the last column. By the factorization lemma for linear maps, we deduce that there exists a (unique) $L_1 \in M_{1,n-2}(\mathbb{K})$ such that every matrix M of $R(W)$ has $\begin{bmatrix} C_M \\ L_1 C_M \end{bmatrix}$ as the last column (for some $C_M \in M_{n-2,1}(\mathbb{K})$).

Claim 1. For $Q := \begin{bmatrix} I_{n-2} & 0 \\ -L_1 & 1 \end{bmatrix}$, one has $QR(W)Q^{-1} = \text{NT}_{n-1}(\mathbb{K})$.

Proof. Set $\mathcal{T} := QR(W)Q^{-1}$. With the above results on $R(W)$, we find:

- (a) Every matrix of \mathcal{T} has entry 0 at the $(n-1, n-1)$ -spot and, for every $C_1 \in M_{n-2,1}(\mathbb{K})$, the subspace \mathcal{T} contains a matrix with $\begin{bmatrix} C_1 \\ 0 \end{bmatrix}$ as the last column;
- (b) For every $N \in \text{NT}_{n-2}(\mathbb{K})$, the subspace \mathcal{T} contains a matrix of the form $\begin{bmatrix} N & 0 \\ ? & 0 \end{bmatrix}$.

Moreover, \mathcal{T} is a $\binom{n-1}{2}$ -dimensional nilpotent subspace of $M_{n-1}(\mathbb{K})$. By Gerstenhaber's theorem, there is an increasing sequence $\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n-1} = \mathbb{K}^{n-1}$ of linear subspaces such that $\forall k \in \llbracket 1, n-1 \rrbracket$, $\forall M \in \mathcal{T}$, $M(E_k) \subset E_{k-1}$. Point (a) then yields $E_{n-2} = \mathbb{K}^{n-2} \times \{0\}$. It follows that every matrix of \mathcal{T} has 0 as the last row, hence point (b) may be refined as follows:

- (b') For every $N \in \text{NT}_{n-2}(\mathbb{K})$, the subspace \mathcal{T} contains $\tilde{N} := \begin{bmatrix} N & 0 \\ 0 & 0 \end{bmatrix} \in M_{n-1}(\mathbb{K})$.

However $\mathbb{K}^{n-3} \times \{0\} \subset \sum_{N \in \text{NT}_{n-2}(\mathbb{K})} \tilde{N}(E_{n-2}) \subset E_{n-3}$, and $\dim(\mathbb{K}^{n-3} \times \{0\}) = \dim E_{n-3}$, therefore $E_{n-3} = \mathbb{K}^{n-3} \times \{0\}$. Continuing by downward induction, we find that $E_k = \mathbb{K}^k \times \{0\}$ for every $k \in \llbracket 0, n-1 \rrbracket$, which shows that $\mathcal{T} \subset \text{NT}_{n-1}(\mathbb{K})$, and hence $\mathcal{T} = \text{NT}_{n-1}(\mathbb{K})$ since $\dim \mathcal{T} = \binom{n-1}{2} = \dim \text{NT}_{n-1}(\mathbb{K})$. \square

We now set $Q_1 := \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} \in \text{GL}_n(\mathbb{K})$ and replace V with $Q_1 V Q_1^{-1}$: again, condition (1) still holds in this new setting, and $J(Z)$ has not been modified, therefore condition (2) still holds. Since Q_1 stabilizes $\{0\} \times \mathbb{K}^{n-1}$, the subspaces W and $R(W)$ have been replaced respectively with $Q_1 W Q_1^{-1}$ and $Q R(W) Q^{-1}$, therefore we now have:

$$R(W) = \text{NT}_{n-1}(\mathbb{K}). \quad (3)$$

3.2 Special matrices in V

Using conditions (1) and (2), we find that, for every $N \in \text{NT}_{n-1}(\mathbb{K})$, the subspace V contains a unique matrix of the form $\begin{bmatrix} N & 0 \\ ? & 0 \end{bmatrix}$. We deduce that:

- There are two linear maps $\varphi : M_{1,n-2}(\mathbb{K}) \rightarrow M_{1,n-2}(\mathbb{K})$ and $f : M_{1,n-2}(\mathbb{K}) \rightarrow \mathbb{K}$ such that, for every $L \in M_{1,n-2}(\mathbb{K})$, the subspace V contains the matrix

$$A_L := \begin{bmatrix} 0 & L & 0 \\ 0 & 0_{n-2} & 0 \\ f(L) & \varphi(L) & 0 \end{bmatrix};$$

- There is a linear form $h : \text{NT}_{n-2}(\mathbb{K}) \rightarrow \mathbb{K}$ such that, for every $U \in \text{NT}_{n-2}(\mathbb{K})$, the subspace V contains the matrix

$$E_U := \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ h(U) & 0 & 0 \end{bmatrix}$$

(there, we have also used condition (3)).

Since $\dim C(V) = n$, we know that some matrix of V has entry 1 at the $(1, n)$ -spot. By linearly combining such a matrix with I_n and a well-chosen A_L , we deduce that every row matrix $L' \in M_{1,n}(\mathbb{K})$ is the first row of some matrix of V . Denote by G the linear subspace of V consisting of the matrices $M \in V$ with $\text{tr } M = 0$ and all columns zero starting from the second one. Applying the rank theorem then shows that $\dim V = n + \dim R(W) + \dim G$. However $n + \dim R(W) = n + \binom{n-1}{2} = \dim V$, therefore $G = \{0\}$.

Condition (3) then yields that, for every $N \in \text{NT}_{n-1}(\mathbb{K})$, the linear subspace V contains a unique matrix of the form $\begin{bmatrix} 0 & 0 \\ ? & N \end{bmatrix}$. We deduce a new family of matrices in V :

- There are two linear maps $\psi : M_{n-2,1}(\mathbb{K}) \rightarrow M_{n-2,1}(\mathbb{K})$ and $g : M_{n-2,1}(\mathbb{K}) \rightarrow \mathbb{K}$ such that, for every $C \in M_{n-2,1}(\mathbb{K})$, the subspace V contains the matrix

$$B_C := \begin{bmatrix} 0 & 0 & 0 \\ \psi(C) & 0_{n-2} & C \\ g(C) & 0 & 0 \end{bmatrix}.$$

Finally, we have seen that some matrix of V has $[0 \ \cdots \ 0 \ 1]$ as first row. By adding to it a well-chosen matrix of the form B_C , we find a matrix

$$J = \begin{bmatrix} 0 & 0_{1,n-2} & 1 \\ ? & ? & 0_{n-2,1} \\ ? & ? & ? \end{bmatrix} \in V.$$

Obviously the subspace $\{A_L \mid L \in M_{1,n-2}(\mathbb{K})\} + \{B_C \mid C \in M_{n-2,1}(\mathbb{K})\} + \{E_U \mid U \in \text{NT}_{n-2}(\mathbb{K})\} + \text{span}(J, I_n)$ of V has dimension $2(n-2) + \binom{n-2}{2} + 2 = \dim V$, hence V is spanned by the above matrices. At this point, we need to examine three cases separately: $n \geq 5$, $n = 4$ and $n = 3$ (the last one is dealt with in Section 4).

3.3 The case $n \geq 5$

3.3.1 Analyzing φ and ψ

Claim 2. *There exists $\lambda \in \mathbb{K}$ such that $\varphi = \lambda \text{id}$ and $\psi = -\lambda \text{id}$.*

Notice first that given $(L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K})$, one has $\text{rk}(A_L) \leq 2$ and $\text{rk}(B_C) \leq 2$, and hence $\text{rk}(A_L + B_C) \leq 4$: the matrix $A_L + B_C$, which belongs to V , is singular and therefore nilpotent.

Proof. Let $(L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K})$.

Assume that $LC = 0$. Then

$$(A_L + B_C)^2 = \begin{bmatrix} L\psi(C) & 0 & 0 \\ ? & \psi(C)L + C\varphi(L) & 0 \\ ? & ? & \varphi(L)C \end{bmatrix}.$$

Since $A_L + B_C$ is nilpotent, we deduce that $L\psi(C) = \varphi(L)C = 0$.

We equip $M_{1,n-2}(\mathbb{K})$ with the non-degenerate symmetric bilinear form $(L_1, L_2) \mapsto L_1 L_2^T$.

Let $L \in M_{1,n-2}(\mathbb{K})$. Then the above result yields $\varphi(L) \in (\{L\}^\perp)^\perp = \text{span}(L)$. It follows that every non-zero vector of $M_{1,n-2}(\mathbb{K})$ is an eigenvector of φ , which classically yields that $\varphi = \lambda \text{id}$ for some $\lambda \in \mathbb{K}$.

With the same line of reasoning, we find that $\psi = \mu \text{id}$ for some $\mu \in \mathbb{K}$.

Choose $(L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K})$ such that $LC \neq 0$. Then A_L , B_C and $A_L + B_C$ are all singular, therefore $\text{tr}(A_L B_C) = 0$ by Lemma 11, i.e., $\psi(L)C + L\varphi(C) = 0$. We deduce that $(\lambda + \mu)LC = 0$, and hence $\mu = -\lambda$. \square

3.3.2 One last conjugation

Set $P' := \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ \lambda & 0 & 1 \end{bmatrix} \in \text{GL}_n(\mathbb{K})$, and note that, for every $(L, C, U) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K}) \times \text{NT}_{n-2}(\mathbb{K})$:

$$(P')^{-1}A_L P' = \begin{bmatrix} 0 & L & 0 \\ 0 & 0_{n-2} & 0 \\ f(L) & 0 & 0 \end{bmatrix} \quad ; \quad (P')^{-1}B_C P' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0_{n-2} & C \\ g(C) & 0 & 0 \end{bmatrix}$$

and

$$(P')^{-1}E_U P' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ h(U) & 0 & 0 \end{bmatrix}.$$

Notice that $P'e_n = e_n$ hence $(P')^{-1}VP'$ still satisfies condition (1). The above matrices show that condition (2) and (3) are obviously satisfied. Replacing V with $(P')^{-1}VP'$, we thus preserve all the previous conditions but we now have the additional one:

$$\lambda = 0.$$

At this point, our aim is to prove that $V \subset \mathbb{K}I_n + \text{NT}_n(\mathbb{K})$, which will suffice since V and $\mathbb{K}I_n + \text{NT}_n(\mathbb{K})$ have the same dimension. In order to do so, we prove that every matrix of the type A_L , B_C , E_U or J is strictly upper-triangular: this suffices to prove our claim since these matrices, together with I_n , span V .

3.3.3 Analyzing f , g and h

Claim 3. *One has $f = 0$ and $g = 0$.*

Proof. Let $(L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K})$. Set $\alpha := f(L) + g(C)$. Then a straightforward computation shows that

$$(A_L + B_C)^3 = \begin{bmatrix} \alpha LC & 0 & 0 \\ 0 & \alpha CL & 0 \\ 0 & 0 & \alpha LC \end{bmatrix}.$$

However $A_L + B_C$ is nilpotent (see Paragraph 3.3.1), therefore $\alpha LC = 0$. It follows that $LC \neq 0 \Rightarrow f(L) + g(C) = 0$.

Let $C_1 \in M_{1,n-2}(\mathbb{K})$. Since $n - 2 \geq 2$, we may choose $L \in M_{1,n-2}(\mathbb{K}) \setminus \{0\}$ such that $LC_1 = 0$. Then the linear map g is constant on the affine hyperplane $\{C \in M_{1,n-2}(\mathbb{K}) : LC = 1\}$, therefore g vanishes on its translation vector space $\{C \in M_{1,n-2}(\mathbb{K}) : LC = 0\}$. In particular $g(C_1) = 0$. We deduce that $g = 0$. The same line of reasoning yields $f = 0$. \square

Claim 4. *One has $h = 0$.*

Proof. Let $U \in \text{NT}_{n-2}(\mathbb{K})$ such that $\text{rk } U = 1$. Set $\beta := h(U)$. Note that $U^2 = 0$ since U is a rank 1 nilpotent matrix. Let $(L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K})$. Set

$$M := A_L + B_C + E_U = \begin{bmatrix} 0 & L & 0 \\ 0 & U & C \\ \beta & 0 & 0 \end{bmatrix}.$$

A straightforward computation (using $U^2 = 0$) yields

$$M^3 = \begin{bmatrix} \beta LC & 0 & LUC \\ \beta UC & \beta CL & 0 \\ 0 & \beta LU & \beta LC \end{bmatrix}.$$

Note that $\text{rk } A_L \leq 1$, $\text{rk } B_C \leq 1$ and $\text{rk } E_U \leq 2$, therefore $\text{rk } M \leq 4$. Since $n \geq 5$, we deduce that M is nilpotent. It follows that $LUC = 0 \Rightarrow \beta LC = 0$.

Choosing $L := \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$ and $C := L^T$, we then have $LUC = 0$ (since U is strictly upper-triangular) whilst $LC = 1$, and we deduce that $\beta = 0$.

We have just established that the linear form h vanishes on every rank 1 matrix of $\text{NT}_{n-2}(\mathbb{K})$, which proves our claim since $\text{NT}_{n-2}(\mathbb{K})$ is obviously spanned by its rank 1 matrices. \square

We thus have, for every $(L, C, U) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K}) \times \text{NT}_{n-2}(\mathbb{K})$,

$$A_L = \begin{bmatrix} 0 & L & 0 \\ 0 & 0_{n-2} & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad B_C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0_{n-2} & C \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It now suffices to show that J is strictly upper-triangular.

3.3.4 Dissecting J

Here, we aim at proving that the matrix J of Section 3.2 may be chosen to have zero entries everywhere except at the $(1, n)$ -spot.

Adding to J a well-chosen matrix of type E_U , we may assume that

$$J = \begin{bmatrix} 0 & 0 & 1 \\ C_1 & T & 0 \\ b & L_1 & a \end{bmatrix} \quad \text{where } L_1 \in M_{1, n-2}(\mathbb{K}), C_1 \in M_{n-2, 1}(\mathbb{K}), (a, b) \in \mathbb{K}^2$$

and $T = (t_{i,j})$ is a *lower-triangular* matrix of $M_{n-2}(\mathbb{K})$.

Claim 5. *One has $C_1 = 0$ and $L_1 = 0$, whilst $T = \alpha I_{n-2}$ for some $\alpha \in \mathbb{K}$.*

Proof. Denote by l the last entry of L_1 . Setting $L_2 := [0 \ \cdots \ 0 \ 1]$, we remark that V contains the matrix $M := J + A_{L_2} + (l-a) \cdot I_n + (t_{n-2, n-2} + l - a) B_{L_2^T}$, which has identical $(n-1)$ -th and n -th columns (and the same non-diagonal entries as J on the first $n-2$ rows) and is therefore singular. Choose $U \in \text{NT}_{n-2}(\mathbb{K})$ which has a last column equal to zero. Again, $M + E_U$ is singular (it has the same last two columns as M). So are E_U and M , therefore Lemma 11 yields $\text{tr}(ME_U) = 0$.

By varying U , we deduce that $t_{i,j} = 0$ for every $(i, j) \in \llbracket 1, n-2 \rrbracket^2$ such that $j < i \leq n-3$.

With the same line of reasoning, we find that $\text{tr}(MA_L) = 0$ for every $L \in M_{1, n-2}(\mathbb{K})$ with 0 as the last entry, which shows that C_1 has zero entries from the first to the $(n-3)$ -th one.

With the same line of reasoning, but replacing the last two columns with the first two rows, we find that $t_{i,j} = 0$ for every $(i, j) \in \llbracket 1, n-2 \rrbracket^2$ such that $2 \leq j < i$, and L_1 has zero entries starting from the second one.

Denote by (e_1, \dots, e_n) the canonical basis of \mathbb{K}^n . With the above results, we find that e_3, \dots, e_{n-1} are eigenvectors of J with respective eigenvalues $t_{2,2}, \dots, t_{n-2, n-2}$, therefore $t_{2,2} = \dots = t_{n-2, n-2}$.

Moreover e_2, \dots, e_{n-2} are eigenvectors of J^T with respective eigenvalues $t_{1,1}, \dots, t_{n-3, n-3}$, therefore $t_{1,1} = \dots = t_{n-3, n-3}$. Since $n-2 \geq 3$, it follows that the $t_{i,i}$'s are all equal to some $\alpha \in \mathbb{K}$.

Let us now consider $J' := J - \alpha I_n \in V$. Note that $\text{rk}(J') \leq 3$ since all the columns of J' from the third one to the $(n-1)$ -th one are zero. Let $L \in$

$M_{1,n-2}(\mathbb{K})$. Then $\text{rk}(J' + A_L) \leq 4 < n$. The matrices J' , A_L and $J' + A_L$ are all singular, therefore Lemma 11 shows that $\text{tr}(J'A_L) = 0$ i.e., $LC_1 = 0$. Since this holds for every $L \in M_{1,n-2}(\mathbb{K})$, we find that $C_1 = 0$.

With the same line of reasoning (but using the B_C 's instead), we find $L_1 = 0$.

Finally, with the same line of reasoning with the elementary matrix $U = E_{1,n-2} \in \text{NT}_{n-2}(\mathbb{K})$ with zero entries everywhere except at the $(1, n-2)$ -spot where the entry is 1, we find $\text{tr}(J'U) = 0$ (note that $\text{rk}(U) = 1$), i.e., $t_{n-2,1} = 0$. With the above results, this finally shows that $T = \alpha I_{n-2}$. \square

Remark 2. If $\text{char}(\mathbb{K}) \neq 2$ (still assuming that $\text{char}(\mathbb{K}) \mid n$), then the above proof may be greatly simplified. Indeed, if a matrix of $M_n(\mathbb{K})$ has $\lambda \in \overline{\mathbb{K}}$ as sole eigenvalue, then its characteristic polynomial is $((x - \lambda)^{n/p})^p$, where $p := \text{char}(\mathbb{K})$, hence $c_2(M) = 0$. It follows that c_2 vanishes everywhere on V , and therefore $\text{tr}(MN) = 0$ for every $(M, N) \in V^2$. Applying this to $M = J$ and N being of any one of the types A_L , B_C and E_U , we find that $C_1 = 0$, $L_1 = 0$ and T is diagonal. We let the reader finish the proof in that case.

Claim 6. *One has $a = b = \alpha = 0$.*

Proof. Set $J' := J - \alpha I_n$. Then

$$J' = \begin{bmatrix} -\alpha & 0 & 1 \\ 0 & 0_{n-2} & 0 \\ b & 0 & a - \alpha \end{bmatrix}$$

Then J' is singular, and therefore nilpotent since $J' \in V$. Then $\text{tr}(J') = 0$ and $c_2(J') = 0$, which shows that $a - \alpha = \alpha$ and $b = -\alpha^2$. Therefore

$$J' = \begin{bmatrix} -\alpha & 0 & 1 \\ 0 & 0_{n-2} & 0 \\ -\alpha^2 & 0 & \alpha \end{bmatrix}.$$

Choose $(L, C) \in M_{1,n-2}(\mathbb{K}) \times M_{n-2,1}(\mathbb{K})$ such that $LC = 1$.

Set $M := J' + A_L + B_C = \begin{bmatrix} -\alpha & L & 1 \\ 0 & 0_{n-2} & C \\ -\alpha^2 & 0 & \alpha \end{bmatrix}$. Note that $\text{rk}(M) \leq 3$. Hence M is

singular and therefore nilpotent. A straightforward computation shows that the first column of M^3 is $[-\alpha^2 \ 0 \ \cdots \ 0]^T$, which yields $\alpha = 0$. The conclusion easily follows since $a = 2\alpha$ and $b = -\alpha^2$. \square

Our proof is now complete: we know that J is strictly upper-triangular, and so is any matrix of type A_L , B_C or E_U . It follows that $V \subset \mathbb{K}I_n + \text{NT}_n(\mathbb{K})$, and the equality of spaces follows from the equality of their dimensions. This finishes our proof of Theorem 5.

3.4 The case $n = 4$ and $\text{char}(\mathbb{K}) = 2$

Recall from Theorem 6 the definition of

$$\mathcal{H} = \mathbb{K}I_4 + \left\{ \begin{bmatrix} 0 & l_1 & l_2 & x \\ c_2 & 0 & y & c_1 \\ c_1 & x & 0 & c_2 \\ y & l_2 & l_1 & 0 \end{bmatrix} \mid (l_1, l_2, c_1, c_2, x, y) \in \mathbb{K}^6 \right\},$$

which is obviously a 7-dimensional linear subspace of $M_4(\mathbb{K})$.

Claim 7. *The set \mathcal{H} is a $\overline{1}$ -spec subspace of $M_4(\mathbb{K})$ and it is not similar to $\mathbb{K}I_4 + \text{NT}_4(\mathbb{K})$.*

Proof. A straightforward computation shows that, for every $(l_1, l_2, c_1, c_2, x, y) \in \mathbb{K}^6$, the matrix

$$\begin{bmatrix} 0 & l_1 & l_2 & x \\ c_2 & 0 & y & c_1 \\ c_1 & x & 0 & c_2 \\ y & l_2 & l_1 & 0 \end{bmatrix} \text{ has characteristic polynomial } t^4 + a \text{ (in } \mathbb{K}[t]),$$

with $a = ((l_1 + l_2)(c_1 + c_2) + xy)^2 + l_1c_2(x + y)^2$, and has therefore a unique eigenvalue in $\overline{\mathbb{K}}$ (since $\text{char}(\mathbb{K}) = 2$). It follows that \mathcal{H} is a $\overline{1}$ -spec subspace.

If \mathcal{H} were similar to $\mathbb{K}I_4 + \text{NT}_4(\mathbb{K})$, the set of singular matrices of \mathcal{H} would be

$$\text{a linear subspace of } \mathcal{H}. \text{ However, } \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{ are singular,}$$

whereas their sum is not. Therefore \mathcal{H} is not similar to $\mathbb{K}I_4 + \text{NT}_4(\mathbb{K})$. \square

We now come right back to the end of Section 3.2 and try to prove that V is similar to $\mathbb{K}I_4 + \text{NT}_4(\mathbb{K})$ or to \mathcal{H} . Notice first that, for every $M \in V$, its characteristic polynomial has the form $(t + \lambda)^4 = t^4 + \lambda^4$ for some $\lambda \in \overline{\mathbb{K}}$, therefore c_2 and c_3 vanish everywhere on V .

Claim 8. *There is a (unique) matrix $A \in M_2(\mathbb{K})$ such that $\forall L \in M_{1,2}(\mathbb{K})$, $\varphi(L) = LA$ and $\forall C \in M_{2,1}(\mathbb{K})$, $\psi(C) = AC$.*

Proof. Indeed, we know that there are two matrices A and B in $M_2(\mathbb{K})$ such that $\forall L \in M_{1,2}(\mathbb{K})$, $\varphi(L) = LA$ and $\forall C \in M_{2,1}(\mathbb{K})$, $\psi(C) = BC$. Since c_2 vanishes everywhere on V , we have $\text{tr}(A_L B_C) = 0$ for every $(L, C) \in M_{1,2}(\mathbb{K}) \times M_{2,1}(\mathbb{K})$, which yields $LAC + LBC = 0$, i.e., $L(A + B)C = 0$. It follows that $A + B = 0$, hence $B = A$ since $\text{char}(\mathbb{K}) = 2$. \square

Claim 9. *One has $f = 0$ and $g = 0$.*

Proof. Let $(L, C) \in M_{1,2}(\mathbb{K}) \times M_{2,1}(\mathbb{K})$. Set $\alpha := f(L) + g(C)$ and

$$M := A_L + B_C = \begin{bmatrix} 0 & L & 0 \\ AC & 0 & C \\ \alpha & LA & 0 \end{bmatrix}.$$

Notice that $c_3(M) = \text{tr}(\text{com}(M))$, where $\text{com}(M)$ denotes the comatrix of M . Obviously the first and last diagonal entries of $\text{com}(M)$ are zero hence

$$c_3(M) = \begin{vmatrix} 0 & l_2 & 0 \\ ? & 0 & c_2 \\ \alpha & ? & 0 \end{vmatrix} + \begin{vmatrix} 0 & l_1 & 0 \\ ? & 0 & c_1 \\ \alpha & ? & 0 \end{vmatrix} = \alpha LC,$$

where $L = [l_1 \ l_2]$ and $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Therefore $LC \neq 0 \Rightarrow f(L) + g(C) = 0$. As in the proof of Claim 3, we deduce that $f = 0$ and $g = 0$. \square

Remark that $\text{NT}_2(\mathbb{K})$ is spanned by $K := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Setting now $\alpha := h(K)$

and $E := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{bmatrix}$, we deduce that $V = \{A_L \mid L \in M_{1,2}(\mathbb{K})\} + \{B_C \mid C \in M_{2,1}(\mathbb{K})\} + \text{span}(I_4, E, J)$.

Claim 10. *There exists $\lambda \in \mathbb{K}$ such that $A = \begin{bmatrix} \lambda & \alpha \\ \alpha & \lambda \end{bmatrix}$.*

Proof. Let $(L, C) \in M_{1,2}(\mathbb{K}) \times M_{2,1}(\mathbb{K})$ and set $M := A_L + B_C + E$. Then a straightforward computation shows that

$$c_3(M) = \alpha LC + l_1 (AC)_2 + c_2 (LA)_1,$$

where $(AC)_2$ denotes the second entry of AC and $(LA)_1$ the first one of LA .

Write $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}$. Notice then that $(L, C) \mapsto \alpha LC + l_1 (AC)_2 + c_2 (LA)_1$ is a bilinear form whose matrix in the respective canonical bases of $M_{1,2}(\mathbb{K})$ and $M_{2,1}(\mathbb{K})$ is

$$\alpha I_2 + \begin{bmatrix} a_{2,1} & a_{2,2} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{1,1} \\ 0 & a_{1,2} \end{bmatrix} = \begin{bmatrix} \alpha + a_{2,1} & a_{1,1} + a_{2,2} \\ 0 & \alpha + a_{1,2} \end{bmatrix}.$$

Since c_3 vanishes everywhere on V , it follows that this matrix is zero, which yields $a_{2,1} = a_{1,2} = \alpha$ and $a_{2,2} = a_{1,1}$, as claimed. \square

As in Paragraph 3.3.2, we now replace V with $(P')^{-1}VP'$ where $P' := \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_2 & 0 \\ \lambda & 0 & 1 \end{bmatrix} \in \text{GL}_4(\mathbb{K})$. Again, all the former conditions still hold in that case, and we now have $A = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$.

We finish by analyzing J . By summing it with a well-chosen scalar multiple of E , we lose no generality in assuming that

$$J = \begin{bmatrix} 0 & 0 & 1 \\ C_1 & T & 0 \\ \beta & L_1 & a \end{bmatrix}$$

where $(L_1, C_1) \in M_{1,2}(\mathbb{K}) \times M_{2,1}(\mathbb{K})$, $(a, \beta) \in \mathbb{K}^2$ and T is a lower-triangular matrix of $M_2(\mathbb{K})$.

Since c_2 vanishes everywhere on V , we find that $\text{tr}(JA_L) = \text{tr}(JB_C) = \text{tr}(JE) = 0$ for every $(L, C) \in M_{1,2}(\mathbb{K}) \times M_{2,1}(\mathbb{K})$, which yields $C_1 = 0$, $L_1 = 0$ and $t_{2,1} = \alpha$.

Claim 11. *One has $a = 0$, and there exists $b \in \mathbb{K}$ such that $\beta = b^2$ and $T = \begin{bmatrix} b & 0 \\ \alpha & b \end{bmatrix}$.*

Proof. Write $T = \begin{bmatrix} b & 0 \\ \alpha & c \end{bmatrix}$. Denote by (e_1, e_2, e_3, e_4) the canonical basis of \mathbb{K}^4 .

Note that the endomorphism $X \mapsto JX$ of \mathbb{K}^4 stabilizes both of the subspaces $\text{span}(e_2, e_3)$ and $\text{span}(e_1, e_4)$ and the matrices of the induced endomorphisms

in the respective bases (e_2, e_3) and (e_1, e_4) are T and $\begin{bmatrix} 0 & 1 \\ \beta & a \end{bmatrix}$. Therefore those matrices have the same unique eigenvalue in $\overline{\mathbb{K}}$, which must be b . This yields $b = c$, $a = 0$ and $\beta = b^2$. \square

Claim 12. *One has $b = 0$.*

Proof. Choose $(L, C) \in M_{1,2}(\mathbb{K}) \times M_{2,1}(\mathbb{K})$ such that $LC \neq 0$. A straightforward computation shows that

$$c_3(A_L + B_C + J) = b^2 LC,$$

and hence $b = 0$. \square

We now have

$$V = \left\{ \begin{bmatrix} \lambda & l_1 & l_2 & x \\ \alpha c_2 & \lambda & y & c_1 \\ \alpha c_1 & \alpha x & \lambda & c_2 \\ \alpha y & \alpha l_2 & \alpha l_1 & \lambda \end{bmatrix} \mid (l_1, l_2, c_1, c_2, x, y, \lambda) \in \mathbb{K}^7 \right\}.$$

If $\alpha = 0$, then we readily have $V = \mathbb{K}I_4 + \text{NT}_4(\mathbb{K})$.

Assume finally that $\alpha \neq 0$. Then, for $D := \text{Diag}(1, 1, 1, \alpha)$, a straightforward computation yields $D^{-1}VD = \mathcal{H}$. This completes the proof of Theorem 6.

4 On $\overline{1}$ -spec subspaces of $M_3(\mathbb{K})$ when $\text{char}(\mathbb{K}) = 3$

In this section, we assume that $\text{char}(\mathbb{K}) = 3$.

4.1 Opening remarks

Here, we will use considerations from Witt's theory of quadratic forms (see [6, Chapters VII, VIII and IX]).

Since $\text{char}(\mathbb{K}) = 3$, a matrix $M \in M_3(\mathbb{K})$ has only one eigenvalue in $\overline{\mathbb{K}}$ if and only if its characteristic polynomial has the form $x^3 + \alpha$ for some $\alpha \in \mathbb{K}$, i.e., if and only if $\text{tr}(M) = c_2(M) = 0$. It follows that the $\overline{1}$ -spec linear subspaces of $M_3(\mathbb{K})$ are the totally isotropic subspaces of $\mathfrak{sl}_3(\mathbb{K})$ for the symmetric bilinear form $(A, B) \mapsto \text{tr}(AB)$. Consider the non-degenerate symmetric bilinear form $b(A, B) := \text{tr}(AB)$ on $M_3(\mathbb{K})$, and notice that $\mathbb{K}I_3 + \text{NT}_3(\mathbb{K})$ is a 4-dimensional totally isotropic subspace of it. Since $4 = [9/2]$, it follows that the Witt index

of b is 4. Since $\mathfrak{sl}_3(\mathbb{K}) = \{I_3\}^\perp$ and I_3 is b -isotropic, the hyperbolic inflation theorem yields that all the maximal totally isotropic subspaces of $\mathfrak{sl}_3(\mathbb{K})$ have dimension 4 (which gives us a new proof of Corollary 3 in that case) and Witt's extension theorem shows that these subspaces form a single orbit under the (natural) action of the orthogonal group of $(c_2)_{|\mathfrak{sl}_3(\mathbb{K})}$. This however gives us little information on their orbits under *conjugation*, which is the topic of our investigation.

In our study, it will be quite helpful to think in terms of spaces of linear transformations (rather than solely of matrices):

Definition 4. Let E be a finite-dimensional vector space over \mathbb{K} . Denote by $\text{End}(E)$ its vector space of linear endomorphisms.

A linear subspace V of $\text{End}(E)$ is called a $\bar{1}$ -**spec** subspace when every element of V has a sole eigenvalue in $\bar{\mathbb{K}}$.

Given a basis \mathbf{B} of E (with cardinality n), we denote by $M_{\mathbf{B}}(V)$ the linear subspace consisting of the matrices of $M_n(\mathbb{K})$ representing the elements of V in the basis \mathbf{B} .

In the rest of the section, we set $E := \mathbb{K}^3$.

Definition 5. Let V be a 4-dimensional $\bar{1}$ -spec subspace of $\text{End}(E)$.

A vector $x \in E \setminus \{0\}$ is said to be **good for** V when no element of V has $\text{span}(x)$ as its range.

Proposition 8 thus implies that at least one non-zero vector of E is good for V .

4.2 Finishing the reduction of an arbitrary 4-dimensional $\bar{1}$ -spec subspace

Notation 6. For $\delta \in \mathbb{K}$, we set

$$\mathcal{F}_\delta := \text{span} \left(I_3, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & -\delta & -1 \end{bmatrix} \right)$$

and

$$\mathcal{G}_\delta := \text{span} \left(I_3, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -\delta & 0 \end{bmatrix} \right).$$

Let V be a 4-dimensional $\bar{1}$ -spec subspace of $\text{End}(E)$. We may find a vector $e_3 \in E \setminus \{0\}$ which is good for V .

In Sections 3.1 and 3.2, we have shown that we may find two vectors e_1 and e_2 in E such that $\mathbf{B} = (e_1, e_2, e_3)$ is a basis of E and $M_{\mathbf{B}}(V) = \text{span}(I_3, A_1, B_1, J)$, where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ a & \lambda & 0 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} 0 & 0 & 0 \\ \mu & 0 & 1 \\ b & 0 & 0 \end{bmatrix}$$

and J has the form $\begin{bmatrix} 0 & 0 & 1 \\ ? & ? & 0 \\ ? & ? & ? \end{bmatrix}$. Since $\text{tr}(A_1 B_1) = 0$, we find $\mu = -\lambda$.

As in Paragraph 3.3.2, we may then modify e_1 so as to have $\lambda = \mu = 0$ in the new basis (which we still denote by (e_1, e_2, e_3)).

Replacing J with $J + t.I_3$ for a well chosen $t \in \mathbb{K}$, we find in V a matrix of the form

$$J' = \begin{bmatrix} t & 0 & 1 \\ ? & 0 & 0 \\ ? & ? & ? \end{bmatrix}.$$

Then $\text{tr}(J') = 0$, $\text{tr}(A_1 J') = 0$, $\text{tr}(B_1 J') = 0$ and $c_2(J') = 0$ yield:

$$J' = \begin{bmatrix} t & 0 & 1 \\ -a & 0 & 0 \\ -t^2 & -b & -t \end{bmatrix}.$$

At this point, we need to distinguish between several cases:

- Assume $a \neq 0$. Choose an arbitrary $\gamma \in \mathbb{K} \setminus \{0\}$, and set $\delta := \frac{ab}{\gamma^3}$

and $\mathbf{B}' := (\frac{1}{\gamma} e_1, \frac{a}{\gamma^2} e_2, e_3)$. Then $M_{\mathbf{B}'}(V)$ is spanned by I_3 , $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{bmatrix}, \begin{bmatrix} (t/\gamma) & 0 & 1 \\ -1 & 0 & 0 \\ -(t/\gamma)^2 & -\delta & -(t/\gamma) \end{bmatrix}.$$

- If $t = 0$, then we deduce that $M_{\mathbf{B}'}(V) = \mathcal{G}_\delta$.
- If $t \neq 0$, then, by *choosing* $\gamma = t$, we have $\delta = \frac{ab}{t^3}$ and $M_{\mathbf{B}'}(V) = \mathcal{F}_\delta$.

- Assume $a = 0$ and $b \neq 0$. Then $M_{(e_3, e_1, e_2)}(V)$ is spanned by I_3 , $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ b^{-1} & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -t & -t^2 & -b \\ 1 & t & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Multiplying the last matrix by } -\frac{1}{b} \text{ and}$$

summing it with $\frac{t}{b} \cdot I_3 - \frac{t^2}{b} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ b^{-1} & 0 & 0 \end{bmatrix}$, we find that $M_{(e_3, e_1, e_2)}(V)$ is

$$\text{spanned by } I_3, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ b^{-1} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -(t/b) & 0 & 1 \\ -1/b & 0 & 0 \\ -t^2/b^2 & 0 & t/b \end{bmatrix}, \text{ and we}$$

deduce, as above, that $M_{(e_1, e_2, e_3)}(V)$ is similar to \mathcal{F}_0 (if $t \neq 0$) or to \mathcal{G}_0 (if $t = 0$).

- Assume that $a = b = 0$ and $t \neq 0$. Then

$$M_{(e_1, e_2, e_3 - \frac{e_1}{t})}(V) = \text{span} \left(I_3, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/t^2 & 0 & 0 \end{bmatrix} \right)$$

and hence

$$M_{(e_1, e_2, e_3)}(V) \sim \mathcal{I} := \mathbb{K}I_3 \oplus \left\{ \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ z & 0 & 0 \end{bmatrix} \mid (x, y, z) \in \mathbb{K}^3 \right\}.$$

This shows in particular that \mathcal{F}_0 itself is similar to \mathcal{I} : indeed, for $P =$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \text{ we find that } P^{-1}\mathcal{F}_0P \text{ is spanned by } I_3, \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & -1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and it is thus also spanned by } I_3, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

- Finally, if $a = b = t = 0$, then we readily have $V = \mathbb{K}I_3 + \text{NT}_3(\mathbb{K})$.

We may summarize some of the above results as follows:

Lemma 12. *Let (e_1, e_2, e_3) be a basis of E . Assume that there exists $(a, b, t) \in$*

\mathbb{K}^3 such that $a \neq 0$ and $M_{(e_1, e_2, e_3)}(V)$ is spanned by the matrices I_3 , $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ b & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} t & 0 & 1 \\ -a & 0 & 0 \\ -t^2 & -b & -t \end{bmatrix}$.

If $t \neq 0$, then there exists $(\lambda, \mu) \in (\mathbb{K} \setminus \{0\})^2$ such that $M_{(\lambda, e_1, \mu, e_2, e_3)}(V) = \mathcal{F}_{ab/t^3}$.

If $t = 0$, then, for every $\alpha \in \mathbb{K} \setminus \{0\}$, there exists $(\lambda, \mu) \in (\mathbb{K} \setminus \{0\})^2$ such that $M_{(\lambda, e_1, \mu, e_2, e_3)}(V) = \mathcal{G}_{\alpha^3 ab}$.

Proposition 13. *The subspaces \mathcal{I} and \mathcal{F}_0 are similar.*

4.3 Classification theorems

We have just proven a good deal of the following theorem:

Theorem 14. *Assume $\text{char}(\mathbb{K}) = 3$, and let V be a 4-dimensional $\bar{1}$ -spec linear subspace of $M_3(\mathbb{K})$. Then exactly one of the following statements is true:*

- (i) *V is similar to $\mathbb{K}I_3 + \text{NT}_3(\mathbb{K})$;*
- (ii) *V is similar to \mathcal{F}_δ for some $\delta \in \mathbb{K}$;*
- (iii) *V is similar to \mathcal{G}_δ for some $\delta \in \mathbb{K}$.*

Conversely, a straightforward computation shows that \mathcal{F}_δ and \mathcal{G}_δ are 4-dimensional $\bar{1}$ -spec subspaces of $M_3(\mathbb{K})$, for every $\delta \in \mathbb{K}$.

Notation 7. Denote by $\sigma : x \mapsto x^3$ the Frobenius automorphism of \mathbb{K} , and set $j := \sigma - \text{id}$. Note that j is an endomorphism of the group $(\mathbb{K}, +)$.

Notation 8. We define the relation \sim_3 on \mathbb{K} as follows:

$$x \sim_3 y \stackrel{\text{def}}{\iff} \exists (a, b) \in (\mathbb{K} \setminus \{0\}) \times \mathbb{K} : x = a^3 y + b^3.$$

This is obviously an equivalence relation on \mathbb{K} .

Theorem 15. Assume $\text{char}(\mathbb{K}) = 3$. Let $(\delta, \lambda) \in \mathbb{K}^2$. Then:

$$\mathcal{G}_\delta \sim \mathcal{G}_\lambda \Leftrightarrow \delta \sim_3 \lambda \quad \text{and} \quad \mathcal{F}_\delta \sim \mathcal{F}_\lambda \Leftrightarrow \delta = \lambda \text{ mod. } j(\mathbb{K}).$$

Example 3. Assume \mathbb{K} is an algebraically closed field of characteristic 3. Then j and σ are onto, and therefore there are exactly three similarity classes of 4-dimensional $\bar{\mathbb{I}}$ -spec subspaces of $M_3(\mathbb{K})$: those of $\mathbb{K}I_3 + \text{NT}_3(\mathbb{K})$, $\mathcal{F}_0 \sim \mathcal{I}$ and \mathcal{G}_0 .

Example 4. Assume \mathbb{K} is a finite field of characteristic 3. Then σ is onto and hence \sim_3 has a sole equivalence class, whereas $j(\mathbb{K})$ is a subgroup of index 3 of $(\mathbb{K}, +)$ (since the kernel of j is the prime subfield of \mathbb{K} and has therefore three elements). Therefore, there are five similarity classes of 4-dimensional $\bar{\mathbb{I}}$ -spec subspaces of $M_3(\mathbb{K})$.

In order to completely classify the 4-dimensional $\bar{\mathbb{I}}$ -spec subspaces up to similarity, what remains to prove is Theorem 15 and the uniqueness statement in Theorem 14: we achieve this in the next section.

4.4 The uniqueness statements in Theorem 14, and the proof of Theorem 15

Lemma 16. Let $\delta \in \mathbb{K}$. Then neither \mathcal{F}_δ nor \mathcal{G}_δ is similar to $\mathbb{K}I_3 + \text{NT}_3(\mathbb{K})$.

Proof. In $\mathbb{K}I_3 + \text{NT}_3(\mathbb{K})$, the set of singular matrices is the linear subspace $\text{NT}_3(\mathbb{K})$. It thus suffices to find two singular matrices in \mathcal{F}_δ (respectively, \mathcal{G}_δ) some linear combination of which is non-singular.

The matrices $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{bmatrix}$ are singular and belong both to \mathcal{F}_δ and \mathcal{G}_δ . Choosing $t \in \mathbb{K} \setminus \{0, -\delta\}$, we find that $tA + B$ is non-singular, which completes the proof. \square

We turn to the study of possible similarities between spaces of type \mathcal{F}_δ or \mathcal{G}_δ .

Definition 9. A basis \mathbf{B} of V is called *V-adapted* if there exists $\delta \in \mathbb{K}$ such that $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$ or $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$: notice then that δ is uniquely determined by \mathbf{B} and that only one of the conditions $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$ and $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ holds.

Let V be a 4-dimensional $\bar{1}$ -spec subspace of $\text{End}(E)$ which is not similar to $\mathbb{K}I_3 + \text{NT}_3(\mathbb{K})$. Note that the third vector of a V -adapted basis is always good for V .

Here is our strategy:

- given a vector $e_3 \in E$ which is good for V , determine the spaces of matrices associated to the V -adapted bases with e_3 as the last vector;
- then investigate what happens when the last vector of a V -adapted basis is modified by an “elementary” operation.

The following lemma states that some sort of converse statement of Lemma 12 holds:

Lemma 17. *Let (e_1, e_2, e_3) be a V -adapted basis of E . Assume that there exists $(\lambda, \mu, a, b, t) \in \mathbb{K}^5$ such that $\lambda \neq 0$, $\mu \neq 0$, and $M_{(\lambda, e_1, \mu, e_2, e_3)}(V)$ is spanned by*

$$I_3, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ b & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} t & 0 & 1 \\ -a & 0 & 0 \\ -t^2 & -b & -t \end{bmatrix}.$$

If $t \neq 0$, then $M_{(e_1, e_2, e_3)}(V) = \mathcal{F}_{ab/t^3}$.

If $t = 0$, then $M_{(e_1, e_2, e_3)}(V) = \mathcal{G}_{\alpha^3 ab}$ for some $\alpha \in \mathbb{K} \setminus \{0\}$.

Proof. Let $\delta \in \mathbb{K}$ be such that $M_{(e_1, e_2, e_3)}(V) = \mathcal{F}_\delta$ or $M_{(e_1, e_2, e_3)}(V) = \mathcal{G}_\delta$. Note

$$\text{that } M_{(e_1, e_2, e_3)}(V) \text{ is spanned by } I_3, \begin{bmatrix} 0 & \frac{\lambda}{\mu} & 0 \\ 0 & 0 & 0 \\ \frac{a}{\lambda} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ \frac{b}{\lambda} & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} t & 0 & \lambda \\ -\frac{a\mu}{\lambda} & 0 & 0 \\ -\frac{t^2}{\lambda} & -\frac{b}{\mu} & -t \end{bmatrix},$$

$$\text{and hence by } I_3, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \frac{a\mu}{\lambda^2} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \frac{b}{\lambda\mu} & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \frac{t}{\lambda} & 0 & 1 \\ -\frac{a\mu}{\lambda^2} & 0 & 0 \\ -(\frac{t}{\lambda})^2 & -\frac{b}{\lambda\mu} & -\frac{t}{\lambda} \end{bmatrix}.$$

We deduce that $a\mu = \lambda^2$ and $\delta = \frac{b}{\lambda\mu} = \frac{ab}{\lambda^3}$.

This immediately yields the second result if $t = 0$. If $t \neq 0$, then $\frac{t}{\lambda} = 1$ hence $\lambda = t$, which yields the first result. \square

Proposition 18. *Let $\mathbf{B} = (e_1, e_2, e_3)$ be a V -adapted basis. Let $\delta \in \mathbb{K}$.*

(a) Assume $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$.

- (i) For every $\lambda \in \mathbb{K}$ such that $\lambda \underset{3}{\sim} \delta$, there exists a V -adapted basis \mathbf{B}' with e_3 as the last vector such that $M_{\mathbf{B}'}(V) = \mathcal{G}_\lambda$.*

(ii) For every V -adapted basis \mathbf{B}' with e_3 as the last vector, there exists $\lambda \in \mathbb{K}$ such that $\lambda \sim_3 \delta$ and $M_{\mathbf{B}'}(V) = \mathcal{G}_\lambda$.

(b) Assume $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$.

(i) For every $\lambda \in \mathbb{K}$ such that $\lambda = \delta \bmod j(\mathbb{K})$, there exists a V -adapted basis \mathbf{B}' with e_3 as the last vector such that $M_{\mathbf{B}'}(V) = \mathcal{F}_\lambda$.

(ii) For every V -adapted basis \mathbf{B}' with e_3 as the last vector, there exists $\lambda \in \mathbb{K}$ such that $\lambda = \delta \bmod j(\mathbb{K})$ and $M_{\mathbf{B}'}(V) = \mathcal{F}_\lambda$.

(c) For every $s \in \mathbb{K}$, there is a V -adapted basis of the form $(?, f_2, e_3)$ such that $\text{span}(f_2, e_3) = \text{span}(e_2 + se_1, e_3)$.

Proof. We start with a preliminary computation. Let $(s, u, t) \in \mathbb{K}^3$ and set

$$P := \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ s & u & 1 \end{bmatrix}, \text{ and } V_0 \text{ the subspace of } M_3(\mathbb{K}) \text{ spanned by } I_3, A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} t & 0 & 1 \\ -1 & 0 & 0 \\ -t^2 & -\delta & -t \end{bmatrix}.$$

A straightforward computation shows that P is non-singular and $P^{-1}V_0P$ is spanned by

$$I_3, A, \begin{bmatrix} -s^2 & -us & -s \\ s & u & 1 \\ \delta + s^3 - us & ? & -u + s^2 \end{bmatrix} \text{ and } \begin{bmatrix} t - s & ts + s^2 + u & 1 \\ -1 & -s & 0 \\ ts + s^2 + u - t^2 & ? & -s - t \end{bmatrix}.$$

Adding well-chosen linear combinations of the first two matrices to the last two, we find that $P^{-1}V_0P$ is spanned by

$$I_3, A, \begin{bmatrix} 0 & 0 & -s \\ s & u + s^2 & 1 \\ \delta + s^3 & ? & -u - s^2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 \\ -1 & -t & 0 \\ -t^2 & ? & t \end{bmatrix}.$$

Adding to the third one the product of the fourth one with s , we deduce that $P^{-1}V_0P$ is spanned by

$$I_3, A, \begin{bmatrix} 0 & 0 & 0 \\ 0 & u + s^2 - st & 1 \\ \delta + s^3 - st^2 & -(u + s^2 - st)^2 & -u - s^2 + st \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 \\ -1 & -t & 0 \\ -t^2 & ? & t \end{bmatrix}$$

(the (3,2)-th entry of the third matrix is easily obtained using the fact that this matrix is nilpotent). Letting (g_1, g_2, g_3) be an arbitrary basis of \mathbb{K}^3 and denoting by H the linear subspace of $\text{End}(\mathbb{K}^3)$ such that $M_{(g_1, g_2, g_3)}(H) = V_0$, we then find that $u = st - s^2$ if and only if there exists $(a, b) \in (\mathbb{K} \setminus \{0\})^2$ such that $(a.(g_1 + s.g_3), b.(g_2 + s.g_1 + u.g_3), g_3)$ is H -adapted.

If $u = st - s^2$, then $P^{-1}V_0P$ is spanned by

$$I_3, A, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta + s^3 - st^2 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} t & 0 & 1 \\ -1 & 0 & 0 \\ -t^2 & st^2 - s^3 - \delta & -t \end{bmatrix},$$

(the (3,2)-th entry of the fourth matrix is obtained using the fact that the third and fourth matrices are mutually orthogonal for $(M, N) \mapsto \text{tr}(MN)$) and we deduce that $P^{-1}V_0P = \mathcal{G}_{\delta+s^3}$ if $t = 0$, and $P^{-1}V_0P = \mathcal{F}_{\delta+s^3-s}$ if $t = 1$.

- Proof of (a)(i): Let $s \in \mathbb{K}$. Then the above calculation shows that $M_{(e_1+se_3, e_2+se_1-s^2e_3, e_3)}(V) = \mathcal{G}_{\delta+s^3}$. Using Lemma 12, we deduce that, for every $(s, z) \in \mathbb{K} \times (\mathbb{K} \setminus \{0\})$, there exists a V -adapted basis \mathbf{B}' with e_3 as third vector such that $M_{\mathbf{B}'}(V) = \mathcal{G}_{z^3\delta+(zs)^3}$. This obviously yields point (i).

- Proof of (a)(ii): Let $(f_1, f_2) \in E^2$ be such that (f_1, f_2, e_3) is a V -adapted basis.

Then the set of elements of V which vanish on $\text{span}(e_3)$ must be a 1-dimensional subspace. Given such a non-zero element u , we then have $\text{Im}(u) = \text{span}(f_1, e_3)$.

It follows that $\text{span}(e_1, e_3) = \text{span}(f_1, e_3)$, which proves that $f_1 = a(e_1 + se_3)$ for some $a \in \mathbb{K} \setminus \{0\}$ and some $s \in \mathbb{K}$.

Moreover, u may be chosen so as to have $u(e_2) = e_1$, and there exists $b \in \mathbb{K} \setminus \{0\}$ such that $u(f_2) = b f_1$ and $u(f_1) = b e_3$. Therefore $u(f_2) = ab(e_1 + se_3) = ab u(e_2 + se_1)$, which yields a scalar $\mu \in \mathbb{K}$ such that $f_2 = ab(e_2 + se_1 + \mu e_3)$.

Set now $e'_1 := e_1 + se_3$ and $e'_2 := e_2 + se_1 + \mu e_3$.

Then the preliminary calculations show that $\mu = -s^2$ and $M_{(e_1+se_3, e_2+se_1+\mu e_3, e_3)}(V) = \mathcal{G}_{\delta+s^3}$. Using Lemma 17, we deduce that $M_{(f_1, f_2, e_3)}(V) = \mathcal{G}_\lambda$ for some $\lambda \in \mathbb{K}$ satisfying $\lambda \sim \delta$.

Points (b)(i) and (b)(ii) are deduced from the preliminary computation and Lemmas 12 and 17 in the same fashion by taking $t = 1$.

Point (c) follows directly from the proofs of points (a)(i) and (b)(i). \square

We now examine the effect of a simple change of the last vector of a V -adapted basis.

Lemma 19. *Let (e_1, e_2, e_3) be a V -adapted basis of E . Let $\delta \in \mathbb{K}$ such that $M_{(e_1, e_2, e_3)}(V) = \mathcal{G}_\delta$ (respectively, $M_{(e_1, e_2, e_3)}(V) = \mathcal{F}_\delta$).*

(a) *Assume that $\delta \not\equiv 0 \pmod{j(\mathbb{K})}$ (respectively, $\delta \neq 0 \pmod{j(\mathbb{K})}$).*

Then, for every $s \in \mathbb{K}$, the vector $e_3 + se_2$ is the third one of a V -adapted basis \mathbf{B} such that $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ (respectively, $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$).

(b) *If $\delta = 0$, then $e_3 + e_2$ is the third vector of a V -adapted basis \mathbf{B} such that $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ (respectively, $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$).*

Proof. Set $t := 0$ if $M_{(e_1, e_2, e_3)}(V) = \mathcal{G}_\delta$, and $t := 1$ if $M_{(e_1, e_2, e_3)}(V) = \mathcal{F}_\delta$.

Let $s \in \mathbb{K}$. Set $e'_1 := e_1 - s(t + \delta s)e_2 + \delta se_3$, and $e'_3 := e_3 + se_2$.

Then a straightforward computation shows that $M_{(e'_1, e_2, e'_3)}(V)$ is spanned by the matrices

$$I_3, \begin{bmatrix} -s(t + \delta s) & 1 & s \\ -s - s^2t^2 + s^4\delta^2 & s(t - \delta s) & s^2(t - \delta s) \\ 1 + \delta s^2(t - \delta s) - \delta^2s^3 & -\delta s & -\delta s^2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \delta s + t & 0 & 1 \\ -st^2 + \delta^2s^3 - 1 & \delta s & -st \\ -\delta st - t^2 & -\delta & \delta s - t \end{bmatrix}.$$

Adding well-chosen linear combinations of I_3 and the third matrix to the second and the fourth ones, we deduce that $M_{(e'_1, e_2, e'_3)}(V)$ is spanned by the matrices

$$I_3, A_1 = \begin{bmatrix} 0 & 1 & s \\ -s - s^2t^2 + s^4\delta^2 & -st & 0 \\ 1 - \delta^2s^3 & -\delta s & st \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 0 & 0 & 1 \\ -st^2 + \delta^2s^3 - 1 & -t & 0 \\ -t^2 & -\delta & t \end{bmatrix}.$$

By $A_1 \leftarrow A_1 - sA_3$, we deduce that $M_{(e'_1, e_2, e'_3)}(V)$ is spanned by the matrices

$$I_3, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 - \delta^2s^3 + st^2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \delta & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 1 \\ -1 + \delta^2s^3 - st^2 & -t & 0 \\ -t^2 & -\delta & t \end{bmatrix}.$$

Set $\gamma := \delta(1 - \delta^2s^3 + st^2)$ and notice that $\gamma = \delta + (-\delta s)^3$ if $t = 0$, and $\gamma = \delta + j(-\delta s)$ if $t = 1$.

In any case, the assumptions of point (a) show that $\gamma \neq 0$ and therefore $1 - \delta^2s^3 + st^2 \neq 0$: Lemmas 12 and 17 thus yield a V -adapted basis \mathbf{B} with e'_3 as third vector such that $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ if $t = 0$, and $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$ if $t = 1$.

Assume now that $\delta = 0$ and $s = 1$. Then $1 - \delta^2s^3 + st^2 \neq 0$, hence Lemma 12 yields point (b). \square

Lemma 20. *Let (e_1, e_2, e_3) be a V -adapted basis of E . Let $\delta \in \mathbb{K}$ such that $M_{(e_1, e_2, e_3)}(V) = \mathcal{G}_\delta$ (respectively, $M_{(e_1, e_2, e_3)}(V) = \mathcal{F}_\delta$), and assume that $\delta \neq 0$. Then e_2 is the third vector of a V -adapted basis \mathbf{B} such that $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ (respectively, $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$).*

Proof. Set $\mathbf{B} := (e_3, e_1, e_2)$, and $t := 0$ if $M_{(e_1, e_2, e_3)}(V) = \mathcal{G}_\delta$, and $t := 1$ if $M_{(e_1, e_2, e_3)}(V) = \mathcal{F}_\delta$.

Then a straightforward computation shows that $M_{(e_3, e_1, e_2)}(V)$ is spanned by

$$I_3, A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & \delta & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} -t & -t^2 & -\delta \\ 1 & t & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Setting $A'_2 := \frac{1}{\delta} A_2$ and then $A'_1 := A_1 - A'_2$, we find by $M_{(e_3, e_1, e_2)}(V)$ is spanned by

$$I_3, A'_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\delta^{-1} & 0 & 0 \end{bmatrix}, A'_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \delta^{-1} & 0 & 0 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} -t & -t^2 & -\delta \\ 1 & t & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Setting finally $A'_3 := -\delta^{-1}(A_3 - tI_3 + t^2 A'_2)$, we find that $M_{(e_3, e_1, e_2)}(V)$ is spanned by I_3, A'_1, A'_2 and the matrix

$$A'_3 = \begin{bmatrix} -\frac{t}{\delta} & 0 & 1 \\ -\delta^{-1} & 0 & 0 \\ -(\frac{t}{\delta})^2 & \delta^{-1} & \frac{t}{\delta} \end{bmatrix}.$$

Assume $t = 0$. Notice that $\delta = (-\delta)^3 \times (-\delta^{-1}) \times \delta^{-1}$, hence Lemma 12 shows that e_2 is the third vector of a basis \mathbf{B} such that $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$.

Assume $t = 1$. Notice that $\frac{1}{(-t/\delta)^3}(-\delta^{-1}) \times \delta^{-1} = \delta$, hence Lemma 12 shows that e_2 is the third vector of a basis \mathbf{B} such that $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$. \square

We are now ready to conclude. Let us first draw a corollary from the two previous lemmas:

Corollary 21. *Let (e_1, e_2, e_3) be a V -adapted basis of E . Let $\delta \in \mathbb{K}$ such that $M_{(e_1, e_2, e_3)}(V) = \mathcal{G}_\delta$ (respectively, $M_{(e_1, e_2, e_3)}(V) = \mathcal{F}_\delta$), and assume that $\delta \neq 0$. Then, for any non-zero vector z of $\text{span}(e_2, e_3)$, there is a V -adapted basis \mathbf{B} with z as the last vector such that $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ (respectively, $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$).*

This uses of course the trivial fact that given an arbitrary basis (f_1, f_2, f_3) of E , one has $M_{(\lambda f_1, \lambda f_2, \lambda f_3)}(V) = M_{(f_1, f_2, f_3)}(V)$ for any $\lambda \in \mathbb{K} \setminus \{0\}$.

We now prove Theorem 15 and the fact that an \mathcal{F}_δ -space is never similar to a \mathcal{G}_λ -space. First, the implications “ \Leftarrow ” in Theorem 15 both follow directly from points (a)(i) and (b)(i) of Proposition 18.

For the converse implications, let V be a 4-dimensional $\bar{1}$ -spec linear subspace of $\text{End}(E)$, where $E = \mathbb{K}^3$, and assume that there exists a V -adapted basis (e_1, e_2, e_3) with $M_{(e_1, e_2, e_3)}(V) = \mathcal{G}_\delta$ (respectively, $M_{(e_1, e_2, e_3)}(V) = \mathcal{F}_\delta$). Of course, we may assume that $\delta \not\sim_3 0$ (respectively, $\delta \notin j(\mathbb{K})$).

Using point (c) of Proposition 18 together with Corollary 21, we find that every vector of $E \setminus \text{span}(e_1, e_3)$ is the third vector of a V -adapted basis \mathbf{B} for which $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ (respectively, $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$). Choose such a basis (e'_1, e'_2, e'_3) . Then every vector of $E \setminus \text{span}(e'_1, e'_3)$ is the third vector of a V -adapted basis \mathbf{B} for which $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ (respectively, $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$). However $\text{span}(e'_1, e'_3) \cap \text{span}(e_1, e_3) = \text{span}(y)$ for some $y \in E \setminus \{0\}$ (because $e'_3 \notin \text{span}(e_1, e_3)$ and $\dim E = 3$).

Therefore:

(P) : Every vector of $E \setminus \text{span}(y)$ is the third vector of a V -adapted basis \mathbf{B} for which $M_{\mathbf{B}}(V) = \mathcal{G}_\delta$ (respectively, $M_{\mathbf{B}}(V) = \mathcal{F}_\delta$).

By Lemma 17, for every V -adapted basis \mathbf{B} with a third vector which is linearly independent from y , there exists $\lambda \in \mathbb{K}$ such that $M_{\mathbf{B}}(V) = \mathcal{G}_\lambda$ and $\lambda \sim_3 \delta$ (respectively, $M_{\mathbf{B}}(V) = \mathcal{F}_\lambda$ and $\lambda = \delta \bmod j(\mathbb{K})$).

Assume that y is the third vector of a V -adapted basis (g_1, g_2, y) . If $M_{(g_1, g_2, y)}(V)$ equals \mathcal{F}_λ for some $\lambda \in j(\mathbb{K})$ or \mathcal{G}_λ for some $\lambda \in \sigma(\mathbb{K})$, then Lemma 12 shows that we lose no generality in assuming that $\lambda = 0$, and point (b) in Lemma 19 yields a vector $z \in E \setminus \text{span}(y)$ which is the last vector of a V -adapted basis \mathbf{B}' for which $M_{\mathbf{B}'}(V) = \mathcal{G}_0$ or $M_{\mathbf{B}'}(V) = \mathcal{F}_0$: this would contradict property (P).

Hence $M_{(g_1, g_2, y)}(V)$ equals \mathcal{F}_λ for some $\lambda \in \mathbb{K} \setminus j(\mathbb{K})$ or \mathcal{G}_λ for some $\lambda \in \mathbb{K} \setminus \sigma(\mathbb{K})$. Applying point (a) of Proposition 18, we find a V -adapted basis \mathbf{B}' with a third vector outside of $\text{span}(y)$ such that $M_{\mathbf{B}'}(V) = \mathcal{F}_\lambda$ or $M_{\mathbf{B}'}(V) = \mathcal{G}_\lambda$: with the previous results, we deduce that $M_{\mathbf{B}'}(V) = \mathcal{G}_\lambda$ and $\lambda \sim_3 \delta$ (respectively, $M_{\mathbf{B}'}(V) = \mathcal{F}_\lambda$ and $\lambda = \delta \bmod j(\mathbb{K})$).

We have proven that, for every V -adapted basis, the subspace of matrices represented by V in this basis is \mathcal{G}_λ for some $\lambda \sim_3 \delta$ (respectively, \mathcal{F}_λ for some $\lambda = \delta \bmod j(\mathbb{K})$).

This finishes the proof of Theorem 15.

We have also proved that, for any $\delta \in \mathbb{K} \setminus \sigma(\mathbb{K})$, the space \mathcal{G}_δ is never similar to an \mathcal{F}_λ -space, and, for any $\delta \in \mathbb{K} \setminus j(\mathbb{K})$, the space \mathcal{F}_δ is never similar to a \mathcal{G}_λ -space.

In order to complete the proof of Theorem 14, it only remains to establish the following result:

Lemma 22. *The spaces \mathcal{F}_0 and \mathcal{G}_0 are not similar.*

Proof. We have already shown that \mathcal{F}_0 is similar to the space

$$\mathbb{K}I_3 + \left\{ \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ z & 0 & 0 \end{bmatrix} \mid (x, y, z) \in \mathbb{K}^3 \right\},$$

which obviously contains two linearly independent rank 1 matrices.

We prove that \mathcal{G}_0 does not contain such matrices.

Let $M \in \mathcal{G}_0$ with rank 1. We write $M = \begin{bmatrix} \lambda & x & z \\ -z & \lambda & y \\ x & 0 & \lambda \end{bmatrix}$. Considering the lower

right 2×2 sub-matrix, we deduce from $\text{rk } M \leq 1$ that $\lambda = 0$.

Considering the sub-matrix obtained by deleting the second row and the third

column, we find $x = 0$. It easily follows that $z = 0$, therefore $M = y \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Two rank 1 matrices of \mathcal{G}_0 must therefore be linearly dependent, which shows that \mathcal{G}_0 is not similar to \mathcal{F}_0 . \square

This finishes the proof of Theorem 14. We have therefore completely classified the 4-dimensional $\bar{1}$ -spec subspaces of $M_3(\mathbb{K})$, up to similarity.

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